A MATHEMATICA L MODEL TO DETERMINE OPTIMAL DOSES OF QUANTITATIVELY LIMITED CHEMICAL FERTILISERS

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Abstract: Starting from the differential equation (1) and solving it results in academic dependence of yield on a single fertiliser (relation 2). Generalising in two variables we get the relation (4). We determine the constants of the initial conditions and compare with experimental data (Fig. 1 – Fig. 4). We determine the optimal solutions for the different values of the sum of the two fertilisers (N and P); results (table 2 and fig. 5.) facilitate the reading of the optimal shares of quantitatively limited fertiliser doses.

INTRODUCTION

In a modern agriculture, the yield of any crop largely depends on chemical fertilisers: this is why they should be properly applied and as economically as possible. Since soil supply in potassium is in general good and the potassium influence is less, we will only study the impact of nitrogen (N) and phosphorus (P) fertilisers.

In this paper we establish optimal ratios of nitrogen and phosphorus (N and P) when the total amount of N + P applied per ha is t, i.e. \( N + P = t \). for each given t there is another optimal ratio N: P for the yield (e.g. wheat crop) to be maximal. For this we first need to determine the functional, mathematical relationship between yield and fertiliser doses.

MATERIAL AND METHOD

In everything that follows, x is the nitrogen dose, y is the phosphorus dose, and \( f(x, y) \) is yield as a function of the two variables. The best way to find this dependence is a differential equation. First we look for the function \( f(x) \).

We admit that the yield increase \( \frac{df}{dx} \) is proportional not with the saturation deficit \( a - f(x) \) (which would be a too weak saturation), but with \( a^2 - f^2(x) \), i.e.:

\[
\frac{df}{dx} = K_1 (a^2 - f^2)
\]

where \( a \) is maximum yield of saturation, and \( K_1 \) is a proportionality constant.

The relation (1) is a differential equation that can be solved as follows:

\[
\frac{df}{a^2 - f^2} = K_1 dx
\]

\[
\int \frac{df}{a^2 - f^2} = K_1 \int dx + C_1
\]
\[-\frac{1}{2a} \ln \frac{a-f}{a+f} = K_1 x + C_1\]

resulting in:

\[f(x) = a \, th \left[ K_1(x + x_0) \right]\] (2)

\(th\) \(x\) is the hyperbolic tangent function that can be expressed as:

\[th\,x = \frac{1-e^{-2x}}{1+e^{-2x}}\]

The relation (2) is the solution to the equation (1), and it represents the dependency of the yield on the nitrogen dose \(x\) (on N).

Likewise, there is yield dependence on the second fertiliser \(y\) (on P), if it were applied alone:

\[f(y) = b \, th \left[ K_2(y - y_0) \right]\] (3)

In case both fertilisers are applied, yield \(f(x,y)\) is a combination of the formulas (2) and (3), such as:

\[f(x,y) = a \, th \left[ K_1(x + x_0) \right] + b \, th \left[ K_2(y + y_0) \right] + c \left\{ 1 - th \left[ K_1(x + x_0) \right] th \left[ K_2(y + y_0) \right] \right\}\] (4)

The first two terms of the relation (4) is the separate contribution of the two fertilisers to the yield, and the last term is the effect of interaction.

We assumed that this effect decreases with the saturation degree of the fertiliser doses. Academically, when \(x\) and \(y\) tend to the infinite, the two hyperbolic tangents tend to 1 and the last term becomes 0.

The constants in (4) can be determined from the initial conditions taking into account experimental data.

We assume that the sum of the amounts of fertilisers is limited, i.e.:

\[x + y = t\ , \text{from where} \quad y = t - x\]

Then the relation (4) becomes:

\[f(x,t) = a \, th \left[ K_1(x + x_0) \right] + b \, th \left[ K_2(t - x + y_0) \right] + c \left\{ 1 - th \left[ K_1(x + x_0) \right] th \left[ K_2(t - x + y_0) \right] \right\}\] (5)

The maximum of the function (5) can be found by annulling the derivatives in relation with \(x\):

\[
\frac{df}{dx} = \frac{aK_1}{ch^2[K_1(x + x_0)]} - \frac{bK_2}{ch^2[K_2(t - x + y_0)]} - \frac{cK_2 th[K_2(t - x + y_0)]}{ch^3[K_1(x + x_0)]} + \frac{cK_2 th[K_1(x + x_0)]}{ch^3[K_2(t - x + y_0)]} = 0\] (6)

where \(ch\,x\) is the function hyperbolic cosines and has the form:

\[ch\,x = \frac{e^x + e^{-x}}{2}\]
The solution of the equation (6) gives the maximum value for the yield \( f(x,y) \) and for a given \( t \). For another value of \( t \) we get another optimal solution, etc.

RESULTS AND DISCUSSIONS

Experimental data presented in Table 1 show average yields over 15 years (1985-1999) in winter wheat on the brown mollic clayish-illuvial soil at Sânandrei (Timiş County), communicated by IRINA ȚĂRĂU et al. 2002.

<table>
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<tr>
<th>P</th>
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<td>4379</td>
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<tr>
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<td>5052</td>
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<tr>
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<td>4265</td>
<td>4837</td>
<td>5304</td>
<td>5250</td>
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Determining the constants

If the relation (4) confirms different key values in Table 1, such as:

\[
\begin{align*}
    f(100,0) &= axK_1(100+x_0) + b \times y_0 + c \left(1 - \frac{1}{[K_1(x_0 + 100)] K_2 y_0}\right) = 4155, \\
    y_0 &= 30
\end{align*}
\]

we get the following constant values:

\[
\begin{align*}
    a &= 4000; \quad b = 1400; \quad c = 150 \\
    x_0 &= 36.82; \quad y_0 = 30 \\
    K_1 &= 0.01159; \quad K_2 = 0.01
\end{align*}
\]

A graphical representation of the function (4), with the found constant values and with experimental data in Table 2, will look like the graphs in fig. 1, 2, 3, and 4.

![Graph 1](image1.png)  
*Fig. 1. Yield depending on N: P = 0*  
— Academic curve; • experimental data

![Graph 2](image2.png)  
*Fig. 2. Yield depending on N: P = 50*  
— Academic curve; • experimental data
These graphs show a good concordance between academic curves (continuous lines) and experimental data (dotted lines).

**Determining optimal solutions**

We give total amount of fertilisers \( t \) different values and we solve the equation (7) each time; results are shown in Table 2 and are represented graphically in fig. 5.

<table>
<thead>
<tr>
<th>( t = x + y )</th>
<th>( x, N )</th>
<th>( y, P )</th>
<th>( x % )</th>
<th>( y % )</th>
<th>( f_{\text{max}} )</th>
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<td>158</td>
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</table>
CONCLUSIONS

The dependence of yield on fertiliser doses can best be represented with the help of hyperbolic functions.

Table 2 and Figure 5 show that, for any given t, we get optimal doses of nitrogen (N) and phosphorus (P) in kg/ha or percentage, as well as maximum yield.

Example: $t = 150 \Rightarrow N = 108 \text{ kg/ha}; N = 72.0\%; P = 42 \text{ kg/ha}; P = 28.0\%$

Table 2 and Figure 5 also show that P is in optimal share starting from a $t$ (sum of N and P) of 75 kg/ha; at the beginning it is smaller, but it increases up to 45%.

The constants of the relation (4) are not universal, since they depend on each crop apart (and even on the cultivar or the hybrid), on soil and on weather conditions, even if experimental data are averages for periods of 15 years.
LITERATURE